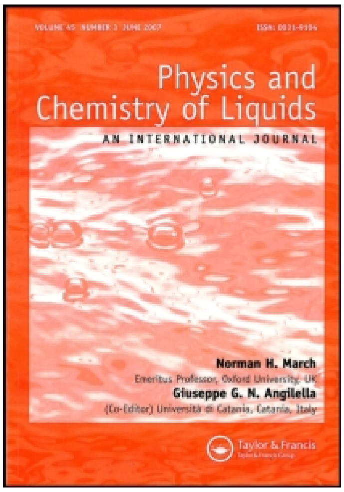

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Collective excitations in a confined, interacting dilute Bose-condensed fluid

G. G. N. Angilella[a]; N. H. March[bc]; R. Pucci[a]
[a] Dipartimento di Fisica e Astronomia, Università di Catania, and Istituto Nazionale per la Fisica della Materia, UdR Catania, Via S. Sofia, I-95123 Catania, Italy [b] Oxford University, Oxford, UK [c] Department of Physics, University of Antwerp, Groenenborgerlaan 171, B-2020 Antwerp, Belgium

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Collective excitations in a confined, interacting dilute Bose-condensed fluid

G.G.N. ANGILELLA*†, N.H. MARCH‡§ and R. PUCCI†

† Dipartimento di Fisica e Astronomia, Università di Catania, and Istituto Nazionale per la Fisica della Materia, UdR Catania, Via S. Sofia, 64, I-95123 Catania, Italy
‡ Oxford University, Oxford, UK
§ Department of Physics, University of Antwerp, Groenenborgerlaan 171, B-2020 Antwerp, Belgium

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In the Thomas–Fermi (TF) regime, S. Stringari [*Phys. Rev. Lett.*, **77**, 2360 (1996)] established the hydrodynamic normal modes of an interacting dilute Bose-condensed fluid confined in an isotropic harmonic trap. Here, we extend this treatment beyond the TF condensate using the work of B. Hu, G. Huang and Y. Ma [*Phys. Rev. A*, **69**, 063608 (2004)] as a starting point. Both numerical results for low-lying eigenfrequencies, for various angular momentum quantum numbers ℓ and $n=0$, and also samples of corresponding eigenfunctions, are presented. Finally, eigenfrequencies are also given for some collective excitations in an axially symmetrical trap.

Keywords: Interacting Bose-condensed fluids; Collective excitations; Spherical and axially symmetrical traps

1. Background and outline

The experimental discovery of Bose–Einstein condensation (BEC) of alkali-metal fluids in magnetic traps has mainly focussed on the physical properties of a confined, interacting, dilute Bose fluid. The widely used Thomas–Fermi (TF) approximation is sufficient to provide a useful description of the interior of the condensate but the surface of the condensate is not amenable to such a treatment. To include the contribution from this surface, Fetter and Feder [1] in an important study calculated the leading correction of the condensate wave function, condensate energy, and low-lying collective excitations beyond the TF description.

The present study has been stimulated by a treatment of Hu, Huang and Ma [2], who have proposed a method for finding analytical solutions of the Bogoliubov-de Gennes (BdG) equations for the low-lying collective excitations in a harmonically

*Corresponding author. Email: Giuseppe.angilella@ct.infn.it

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trapped BE condensate beyond the TF limit. Fetter and Feder [1] explicitly extend the boundary-layer formalism of Dalfovo *et al.* [3] in their study, while Hu *et al.* [2] draw on a variational principle to the approximate ground state of the condensate with a Fetter-like variational ground-state wave function [4,5]. This leads them [2] eventually to a fuller study of the elementary excitations on such trapped, interacting Bose fluids.

The outline of the present article is then as follows. In section 2 we briefly summarize the expressions of the eigenfrequencies, denoted $\bar{\omega}_{n_r\ell}$ following Hu *et al.* [2]. For the special case $n_r = 0$ and $\ell \geq 1$, their complete expression for $\bar{\omega}_{n_r\ell}$ can be in terms of Euler's beta function $B(a, b)$ as given in equation (2.2). Their expression for $\bar{\omega}_{n_r\ell}$ is thoroughly analyzed in section 2, both analytically in limiting cases and using numerical plots. Then, section 3 is concerned with sample results for corresponding eigenfunctions.

While the discussion of sections 2 and 3 is relevant to isotropic harmonic confinement, two Appendices deal, albeit briefly, with the case of axial symmetry. Some approximate results of Hu *et al.* [2] for the eigenfrequencies are proposed. Section 4 constitutes a summary plus a discussion of directions for further work in this important area of many-body physics: namely collective excitations.

2. Low-lying collective excitations with spherical trapping

We have just summarized on the nature of the input in the study of collective eigenfrequencies $\bar{\omega}_{n_r\ell}$ by Hu *et al.* [2]. Their general expression for $\bar{\omega}_{n_r\ell}$ reduced for $n_r = 0$ and $\ell \geq 1$ to the form as in [2]

$$\bar{\omega}_{0\ell} = \frac{(\ell + \ell q)^{1/2}}{2B(\ell + 3/2, q+1)} \left\{ B\left(\ell + \frac{3}{2}, q+1\right) + B\left(\ell + \frac{3}{2}, 2q+1\right) + \frac{\zeta^2}{q}\left[(\ell q^2 - \ell q - 3q) B\left(\ell + \frac{3}{2}, q-1\right) - 2q(1-q)B(\ell+1, q-2)\right]\right\}, \quad (2.1)$$

where Euler's beta function $B(a, b)$ introduced already in section 1 is defined in terms of Euler's gamma function by

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \quad (2.2)$$

The change made in equation (2.1) from the result of Hu *et al.* [2] is to introduce ζ^2/q as one 'independent' variable, i.e. $\bar{\omega}_{0\ell} = \bar{\omega}_{0\ell}(\zeta^2/q, q)$.

The important parameter ζ entering the eigenfrequencies of the collective modes is defined by Hu *et al.* [2] as

$$\zeta = \frac{\hbar\omega_\perp}{2\mu}. \quad (2.3)$$

Here, the trapping potential has the form

$$V_{\text{ext}}(\mathbf{r}) = \frac{1}{2} m\omega_\perp^2 (x^2 + y^2 + z^2), \quad (2.4)$$

where the grand canonical Hamiltonian density of a trapped dilute Bose fluid with a weak repulsive interaction between atoms takes the form

$$\mathcal{H} = -\frac{\hbar^2}{2m}\nabla^2 + V_{\mathrm{ext}}(\mathbf{r}) - \mu + \frac{g}{2}\hat{\Psi}^{\dagger}(\mathbf{r}, t)\hat{\Psi}(\mathbf{r}, t). \tag{2.5}$$

Here $\hat{\Psi}(\mathbf{r}, t)$ is the field operator, $g = 4\pi\hbar^2 a_{sc}/m$ is the constant determining the strength of the atomic interactions, with m as the atomic mass. Finally, a_{sc} (>0) is the s-wave scattering length, while μ denotes the chemical potential of the assembly.

Having established the theoretical input required to characterize the eigenfrequencies for the low-lying collective excitations in such a trapped Bose-condensed fluid, we turn to the numerical evaluation of $\bar{\omega}_{0\ell}$ in equation (2.1). Before presenting plots of $\bar{\omega}^2_{0\ell}$ *versus* ζ^2/q, from which will emerge a valuable choice of one 'independent' variable, let us relate the beta function contributions entering equation (2.1) by noting first that

$$B(a, q-1) = \frac{\Gamma(a)\Gamma(q-1)}{\Gamma(a+q-1)} = \frac{a+q-1}{q-1}B(a, q), \tag{2.6}$$

where $B(a, q)$ on the right-hand side of equation (2.6) is determined by equation (2.2).

2.1. *Representation of eigenfrequencies of collective modes for $n_r = 0$*

In figure 1, the results of equation (2.1) for the square $\bar{\omega}^2_{0\ell}$ of the collective mode frequencies are displayed for values of $\ell = 1$–10. What we emphasize, is the dependence on the variable ζ^2/q, displayed explicitly in equation (2.1). While there remains residual dependence of the variational parameter q (see Appendix A), the curves shown prove that $\bar{\omega}^2_{0\ell}$ in the above range of ℓ depend only weakly on q, provided q remains small ($q = 0.001$–0.01 in figure 1). However, refined treatments beyond the work of Hu *et al.* will include higher terms in ζ^2/q than that displayed in equation (2.1).

3. Sample eigenfunctions for zero radial quantum number and different orbital angular momenta

The purpose of this section is to present numerical calculations of sample eigenfunctions related to the eigenfrequencies recorded in figure 1. As shown in the figure, we assume the radial quantum number n_r to be zero, and allow the orbital angular momentum quantum number to span the range from $\ell = 1$ to 10. In the notation of Hu *et al.* [2], the radial part of their eigenfunction labeled $\varphi_-(\mathbf{r})$ is plotted. Thus, omitting the spherical harmonic $Y_{\ell m}(\theta, \phi)$ in equation (9) of [2], we plot

$$R_{\perp}^{3/2}\varphi_-(r) = (2/I_{0\ell})^{1/2}(\zeta\bar{\omega}^{(0)}_{0\ell})^{-1/2}(1-\bar{r}^2)^{(q+1)/2}\bar{r}^{\ell}P_{0\ell}(\bar{r}^2). \tag{3.1}$$

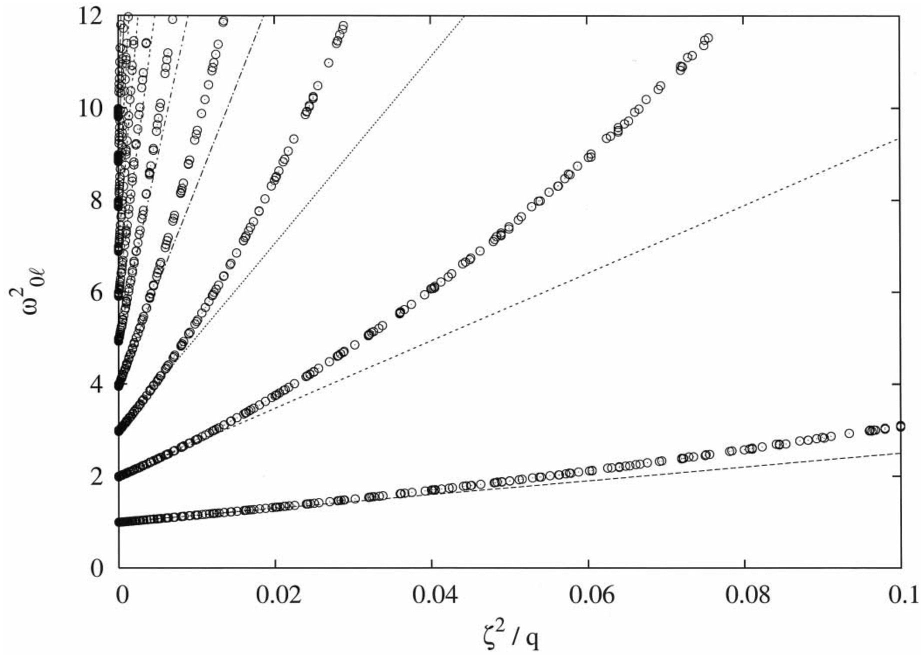

Figure 1. Square $\bar{\omega}_{0\ell}^2$ of the collective mode frequencies, equation (2.1), as a function of ζ^2/q, for $q = 0.001$–0.01 and $\zeta = 0$–0.1, and for different values of $\ell = 1$–10. Despite the fact that $\bar{\omega}_{0\ell}$ in equation (2.1) formally depends on ζ and q separately, the plot shows that, for each value of ℓ, $\bar{\omega}_{0\ell}^2$ almost collapses into a function of the single variable ζ^2/q, at least over the range of values ζ and q considered here. Dashed lines are linear approximations to the square eigenfrequencies in the limits $q \to 0$, $\zeta^2/q \to 0$, corresponding to the TF limit, where $\omega_{0\ell}^2 \to \ell$, as is clearly evident here.

In equation (3.1), $\bar{r} = r/R_\perp$, $P_{0\ell}$ are related to the Jacobi polynomials [2], while the angular frequencies $\bar{\omega}_{0\ell}^{(0)}$ are given explicitly by

$$(\bar{\omega}_{0\ell}^{(0)})^2 = \ell(1+q). \tag{3.2}$$

Finally, $I_{0\ell}$ appearing in the normalization of $\varphi_-(r)$ is defined by

$$I_{0\ell} = \int_0^1 dx\, x^{\ell+1/2}(1-x)^q P_{0\ell}^2(x). \tag{3.3}$$

Figure 2 shows that the eigenfunctions have a maxima which moves outwards towards $R_\perp$ as ℓ increases from 1 to 10. To analyze further how the heights, $\varphi_-^{\max}$, of the curves in figure 2 depend on the orbital angular momentum, we have constructed figure 3 showing, essentially, $\varphi_-^{\max}$ as a function of ℓ. While the dashed curve shown guides the eye passing near the calculated points, the decay with increasing ℓ is approximately fitted by an inverse fractional power of ℓ as recorded in the caption of figure 3.

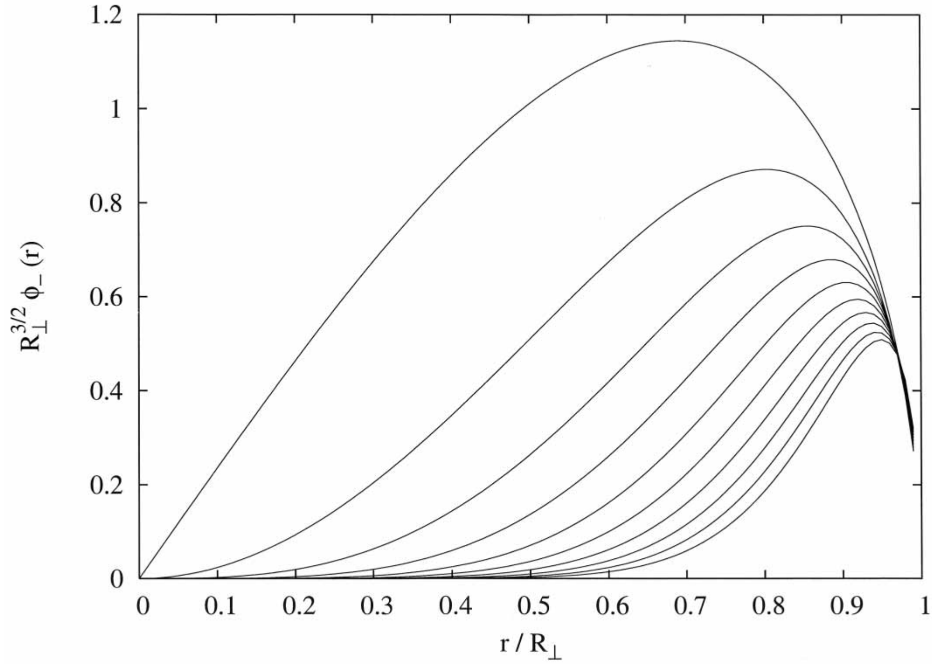

Figure 2. Eigenfunctions $\varphi_-(r)$ for $\ell = 1$–10 (topmost curve: $\ell = 1$), for $q = 0.1$ and $\zeta = 1$.

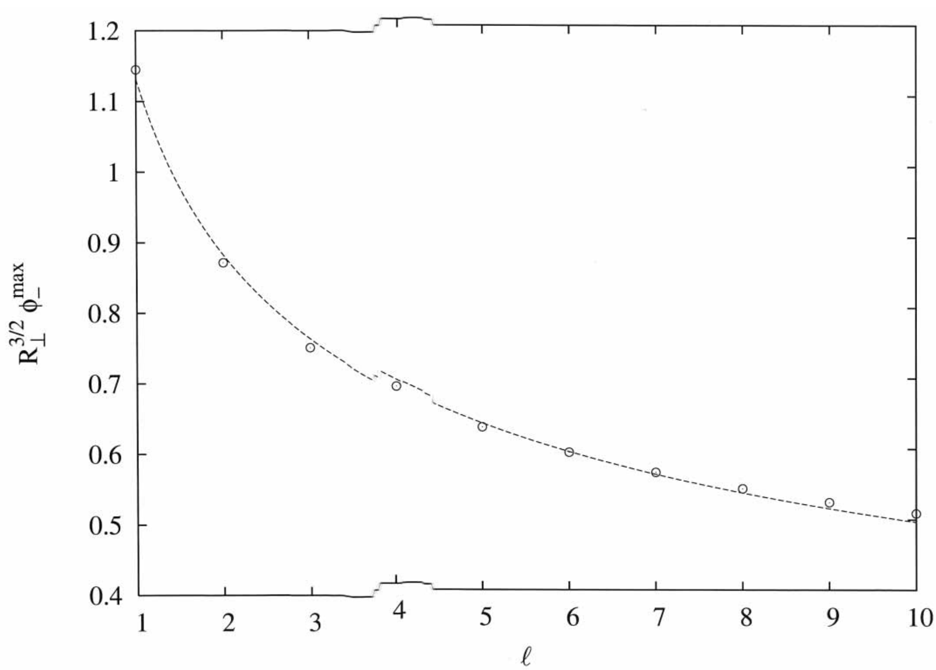

Figure 3. Maximum values of the eigenfunctions $\varphi_-(r)$ *vs.* ℓ. Dashed line is a guide to the eye ($\varphi_-^{\max} \propto \ell^{-0.36}$).

4. Summary and future directions

In addition to the immediate numerical results presented here, especially in figures 2 and 3, our earlier work has intimate connection with the BdG equations [6,7]. Thus, the fact that some analytic progress is possible in solving the BdG equations has motivated us to reopen our own earlier studies, which we hope subsequently to report in detail.

In connection with figures 2 and 3, and especially the plots of low-lying collective excitation modes, further experimental work (presently restricted to a region near the origin of figure 1) will be important in determining the direction of future fruitful theoretical work. One possible refinement is to go beyond the plots in figure 2 which restrict consideration of the ζ^2/q independent variable to $O(\zeta^2/q)$ only. Another, of course is to refine the variational function [4] on which the Hu *et al*. study [2] centres upon.

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Appendices

A. *Route to determine the variational parameter q*

While in figure 1 we demonstrate that ζ^2/q is the dominant independent variable in determining $\bar{\omega}_{0\ell}$ from equation (2.1), there is residual dependence on q. Also, to obtain quantitatively from figure 1 the departures from the TF limiting eigenfrequencies, we must fix ζ^2/q, starting from confining potential parameters defined in equation (2.4) plus the value of the chemical potential μ.

In their study, Hu *et al.* [2] also consider a non-spherically symmetric trap, characterized by a generalized potential given by

$$V_{\text{ext}}(\mathbf{r}) = \frac{1}{2} m\omega_{\perp}^{2}(x^{2} + y^{2} + \lambda^{2}z^{2}). \tag{A.1}$$

In connection with this potential, Ma and Chen [5] constructed the following variational estimate of the energy

$$E_0(R_{\perp}, R_z, q) = \frac{1}{2}\hbar\omega_{\perp}\left[\frac{1}{2}A\left(\frac{2}{R_{\perp}^{2}} + \frac{1}{R_z}\right) + \frac{1}{2}B\left(2R_{\perp}^{2} + \lambda^{2}R_z^{2}\right) + 2\lambda PC\frac{R_{\perp}^{2}}{R_z}\right], \tag{A.2}$$

where q, $R_{\perp}$, R_z are variational parameters, and $R_{\perp}$, R_z are the two radii associated with the axially symmetric trap, with R_z reducing to $R_{\perp}$ in the limit $\lambda = 1$, and

$$P = (N_0 - 1)\frac{a_{sc}}{a_{ho}}, \tag{A.3}$$

N_0 being the particle number in the condensate and $a_{ho} = (\hbar/m\omega_{\perp})^{1/2}$ the characteristic oscillator length of the trapping potential. In equation (A.2), A, B and C are given functions of q, defined as

$$A = \frac{7}{2} + q + \frac{5}{2q} \tag{A.4a}$$

$$B = \left(\frac{7}{2} + q\right)^{-1} \tag{A.4b}$$

$$C = \frac{\Gamma(3 + 2q)}{\Gamma(3/2)\Gamma(2q + 9/2)}\left[\frac{\Gamma(q + 7/2)}{\Gamma(2 + q)}\right]^{2}. \tag{A.4c}$$

Figure A.1 shows P as a function of the variational parameter q, obtained by minimizing the variational estimate of the energy, equation (A.2), with respect to q and $R_{\perp} = R_z$, in the case $\lambda = 1$. Figure A.1 clearly shows that the TF limit is recovered in the limit $q \to 0$ and $P \to \infty$ [2,5].

B. *Approximate expressions for collective mode frequencies $\bar{\omega}_{0\ell}$ in an axisymmetric trap with $\lambda \neq 1$*

Stringari [8] obtained the low energy excitations from the hydrodynamic limit for a harmonically confined Bose-condensed fluid. His result for the frequencies with spherical symmetry may be written as [8]

$$\omega = \omega_{ho}(2n^{2} + 2n\ell + 3n + \ell)^{1/2}, \tag{B.1}$$

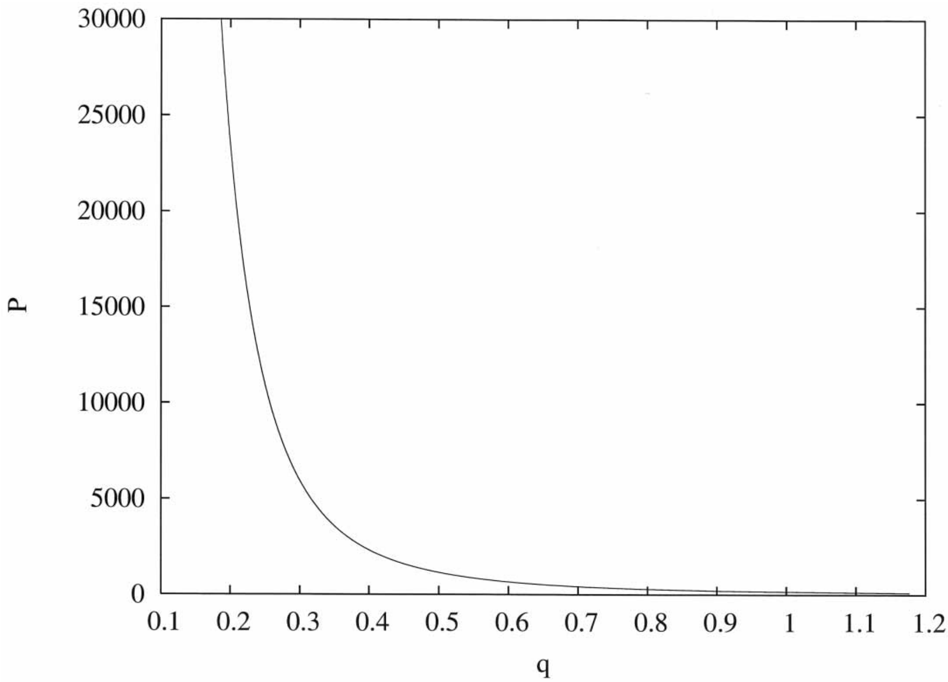

Figure A.1. $P=(N_0-1)a_{sc}/a_{ho}$ as a function of the variational parameter q is shown, determined by numerically minimizing equation (A.2) with respect to q and $R_\perp = R_z$, in the case $\lambda = 1$. The TF limit is clearly recovered when $q \to 0$ and $P \to \infty$.

where ω_{ho} is the angular frequency associated with the harmonic trap. In equation (B.1), n denotes the number of radial nodes, with ℓ throughout giving the orbital angular momentum.

For $\lambda \neq 1$, with

$$V = \frac{1}{2} m\omega_\perp^2 s^2 + \frac{1}{2} m\omega_z^2 z^2, \tag{B.2}$$

where $s=(x^2+y^2)^{1/2}$ is the radial variable in the xy plane. Stringari [8] exploits the axial symmetry to note that the third component m of the angular momentum remains a good quantum number, the excitation frequencies now, however, depending on m. He obtains explicit results in particular cases, which we cite below.

Since available magnetic traps are often highly anisotropic, let us take the case discussed by Stringari [8], who derives a frequency law for two decoupled modes (with $n=1$ and $\ell=0$) of the form [8]

$$\omega^2(m=0) = \omega_\perp^2 \left(2 + \frac{3}{2}\lambda^2 \pm \frac{1}{2}\left(9\lambda^4 - 16\lambda^2 + 16\right)^{1/2} \right), \tag{B.3}$$

with $\lambda = \omega_z/\omega_\perp$. When $\lambda \to 1$, one recovers Stringari's corresponding frequencies $2^{1/2}\omega_{ho}$ and $5^{1/2}\omega_{ho}$ for spherical traps.

As Stringari also stresses, in the limit $\lambda \gg 1$ corresponding to disc-like symmetry, the two solutions in equation (B.3) approach angular frequencies $(10/3)^{1/2}\omega_{\perp}$ and $3^{1/2}\omega_z$, respectively. For the other limit $\lambda \ll 1$ corresponding to cigar-like geometry, the two frequencies are obtained as $(5/2)^{1/2}\omega_z$ and $2\omega_{\perp}$.

These results, of course, bear on the connection between the Hu *et al*. results for $\lambda \neq 1$, and in particular with their spherical trap findings as $\lambda \to 1$.